**'Buy *n* times, get one free' loyalty cards: Are they profitable for competing firms?**

**A game theoretic analysis**[1]

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**Abstract**

This paper evaluates whether firms offering loyalty programs (LPs) should choose a restricted redemption policy by imposing a specific number of purchases before customers can redeem their points. Such restriction is commonly offered in form of 'buy *n* times, get one free' loyalty cards. We develop a Multinomial logit model where consumer's utility depends on the value of the product and of the rewards. Using an iterative algorithm, we numerically solve a Nash game for two firms offering loyalty programs. Optimal strategies and profits are obtained for three different scenarios (games): (1) both firms do not restrict redemption (2) both firms restrict redemption; and (3) only one firm restricts redemption while the other firm does not. Our main findings indicate that each firm's optimal strategies are significantly affected by whether the competitor decides to restrict or not to restrict redemption. In particular, a firm that restricts reward redemption should offer a higher price if its competitor also restricts redemption. Further, the dominant strategy of the game depends on customers' valuations of time and rewards. For example, when customers highly value time but do not highly value rewards, the dominant strategy for both firms is not to restrict redemption. Alternatively, firms can face a Prisoner dilemma situation leading to unrestricted redemption policy for intermediate levels of customer valuation of both time and rewards.

**Keywords:** OR in Marketing, Iterative algorithm, Multinomial logit models, Loyalty programs, Game theory.

## 1. Introduction

Loyalty programs (LPs) are commonly used to help firms strengthen their relationship with their customers through repeat purchase, and ultimately increase their long-term profitability. Recent statistics have reported rapid expansion in the number, size, and scope of LPs, and also in the range of firms and industries that employ them. Between 2000 and 2014 the number of loyalty memberships in the U.S. more than tripled increasing from 0.97 to 3.3 billion (Berry, 2015). Consequently, investments in loyalty programs have been rising steadily growing from $2.7 billion to $8.8 billion between 2000 and 2005 in the US, with a projected annual growth rate of 10.6% (Shevlin, 2009).

In order to reap the benefits that LPs offer, firms choose different designs and features of their programs. The design of LPs is multifaceted, involving several elements, such as reward magnitude, reward type (monetary versus non-monetary), and redemption policies. In particular, some firms commonly impose a pre-defined number of purchasing before consumers can redeem their rewards. Typical examples of such restrictive loyalty programs include coffee shops' reward cards (e.g., 'buy 10 times, get one free'). Such practice has been used by large companies such as Second cup in Canada and Marriott in North America. While these restrictions are commonly used, many firms in the same industries do not impose such restrictions on the redemption of their points. For example, Waves Coffee, a franchised coffee chain in Western Canada, offers its customers loyalty rewards that can be redeemed at any time.

The empirical literature indicates that LPs that restrict reward redemption can have both positive and negative consequences. They can be perceived negatively by customers (Dorotic et al., 2014), may result in lowering customer satisfaction (Stauss et al., 2005), and thereby can be detrimental to profits (Dowling & Uncles, 1997; Noble et al., 2014). However, withholding rewards can also stimulate customer purchases (Kopalle & Neslin, 2003; Drèze & Nunes, 2009). For example, Kivetz et al. (2006) found that for a coffee shop using a 'buy 10 times, get one free' LP, the customers' likelihood to buy a product increases as they approach the reward redemption period. The discrepancy in these empirical results calls for an analytical study that evaluates the implications of such restrictions on redemption policies and identifies conditions under which such restriction could benefit the firms offering the rewards. To our knowledge, such study has not yet been done despite the wide-spread use of both restricted and unrestricted LPs in practice. This paper aims to fill this knowledge gap and to investigate for the first time the profitability of restricting the time of redemption of loyalty rewards. We consider a market formed by two competing firms offering loyalty programs. We develop an analytical model that represents consumers' product and reward redemption choices. Our model is aligned with two important psychological theories: the mental accounting theory, and the goal gradient theory.

The mental accounting theory indicates that customers' valuations of gains and losses differ depending on whether they relate to loyalty rewards or to cash (Thaler, 1985). This is because customers' increased utility from a gain or disutility (pain) from a loss (payment) can vary depending on which currency (cash or rewards) is being exchanged for the payment (Soman, 2003; Drèze & Nunes, 2004). This behavioral theory helps understand customers' decision to redeem or accumulate gains (in cash or in reward points). In fact, at every purchasing occasion, customers have to weigh their gain (either from accumulating points or from receiving a cash discount on price) versus their loss (either from redeemed rewards or from the missed opportunity of price savings).

The goal gradient theory predicts that customers accelerate their purchasing process as they progress towards earning a particular reward (Kivetz et al., 2006). This theory is especially important in understanding consumer redemption behavior for restricted loyalty programs. It indicates that the closer a customer gets to redeeming his/her rewards, the more he/she will feel the pressure to accumulate points, and the more likely that he/she will purchase the product of the firm offering the restricted loyalty program. While these two theories help understand consumers' decisions for redeeming loyalty rewards, they do not offer guidance on the best course of actions for firms designing loyalty programs and considering a restricted versus an unrestricted redemption policy. These insights are also missing from the analytical literature about LPs.

## 2. Literature review

Our paper is connected to several literature streams including the developing operations management literature about the effects of strategic customer behaviour on firms' optimal decisions (e.g., Guo & Hassin, 2012; Aviv & Pazgal, 2008; Besanko & Winston, 1990), and more specifically those that study reward programs (Chevalier & Hirsch, 1982; Gerchak & Kubat, 1986; Pauler & Dick, 2006; Singh et al., 2008; Xiao et al., 2011; Gandomi & Zolfaghari, 2011, 2013). Given the extensive use of loyalty programs in different industries, researchers have recently shown increasing interest in this area. A survey of the existing literature shows that most research to date falls in one of the following two categories: behavioral/empirical studies and analytical studies. Although this paper belongs to the analytical group, it is aligned with important findings obtained in the empirical literature.

The empirical literature consists in descriptive and experimental studies that identify the effects of loyalty programs on consumer attitudes and purchasing behavior (e.g., Devaraj et al., 2001; Buckinx & Van den Poel, 2005; Sousa & Voss, 2009), and provides an understanding of the psychological factors that explain customers' redemption decisions (e.g., Butcher et al., 2001; Bustos-Reyes & González-Benito, 2008). The psychological theories of mental accounting and goal gradient emerge from this literature.

A few recent analytical studies examined the strategic implications of loyalty programs for firms' revenues and profits. These have focused on optimizing the efficiency of loyalty programs using mathematical models that represent simple markets (e.g. Singh et al., 2008; Pauler & Dick, 2006; Gandomi & Zolfaghari, 2011, 2013). For example, the recent study by Gandomi & Zolfaghari (2013) used a two-period model to show that the effects of a loyalty program on a monopolistic firm's revenue depend on customer satisfaction. Chun et al. (2015) developed a multi-period model for a monopolistic firm. A few studies have investigated the efficiency of loyalty programs for competing firms (Klemperer, 1995; Kim et al., 2001; Caminal & Matutes, 1990; Singh et al., 2008). Notably, Singh et al. (2008) used a game-theoretic model to determine the profitability of offering loyalty programs for competing firms where redemption is unrestricted and is offered as percent discount in price. Considering a deterministic Hotelling model and two periods of purchasing, Singh et al. (2008) solved for symmetric and asymmetric equilibrium solutions to show that firms might be better off with no loyalty program even if the competitor is offering one.

The above mentioned papers focused on determining the effectiveness of unrestricted loyalty programs but did not assess the profitability of restricted redemption policies. Further, the mathematical models used in this literature are based on simplistic representations of the market, overlook some important behavioral aspects of loyalty programs, or ignore competitive interactions by focusing on a monopolistic firm. Therefore, an analytical investigation focusing on the profitability of restricted LPs is needed given the conflicting results reported by empirical research on the value of such restrictions. Also, there is a need to develop more comprehensive analytical models about loyalty programs that incorporate the psychological factors influencing consumers' decision making for product purchases and rewards redemption. Such models can enrich our understanding of the efficiency of loyalty programs.

This research fills these literature gaps by evaluating the profitability of loyalty programs with and without restrictive redemption policies for competing firms. We aim to answer the following research questions: For two competing firms offering loyalty programs, which is a more profitable redemption policy: a restricted or an unrestricted one? Under what market conditions? To do so, we develop a comprehensive analytical model based on consumers' valuation of loyalty rewards and reflecting both the mental accounting and goal-gradient theories. In our set-up, two competing firms offer loyalty programs. When the LP is restricted, a specific number of purchases is required before the consumer can redeem the accumulated points to earn a free product. In the unrestricted LP, the reward is a price reduction that the consumer can redeem at any subsequent purchase. We solve a Nash game in three scenarios (games). In the first and second games, both firms choose the same reward redemption policy (both either restrict or not restrict the redemption). In the third game, the firms' policies are asymmetric such as only one firm restricts rewards redemption while the other does not. Comparison of equilibrium outputs across games provides insights into the effects of restricting redemption on the firms' profits. We also examine the effects on firms' revenues and on consumers' redemptions.

The remainder of this paper is organized as follows. Section 3 explains the model formulation, Section 4 describes the methodology used to solve the model, Section 5 presents and discusses the results, and Section 6 summarizes and concludes.

## 3. Model

We first provide a concise model description, followed by a discussion of our modeling choices, consumers' valuation and utility functions, and the firms' demand and profit functions. Table A.1 in the Appendix includes a list of the notations used throughout the paper.

In our model, the market is served by two firms named $a$ and $b$ selling substitutable but similar products. **We assume that the total market size is fixed and is equal to one for simplicity.[3]** To study the effects of restricting redemption, we consider that each firm offers a loyalty program and has the possibility of imposing restrictions on the redemption of rewards earned by its customers**. We assume that the model is static, i.e., the firms play a one-shot game where prices and reward periods are decided only once and are independent of time. This assumption helps focus on understanding the effects of restricting redemption on consumers' choice and therefore firms' demands and profits in isolation of changes in pricing strategies over time.** Further, we assume that firms play a Nash game, i.e., they set their decision variables simultaneously in the beginning of the selling season, without knowing each other's decisions.

Each Firm $i \in \{a, b\}$ chooses its price ($p_i$), and can choose to either restrict or not to restrict redemption of its rewards. This results in three scenarios (games). In the first scenario (S1), both firms choose a restricted redemption policy. In the second scenario (S2), only Firm $a$ restricts redemption while

[33] Note that the total market size does not affect the value of the optimal decision variables. It only affects the size of the demand of each firm, therefore the scale of the firms' revenues and profits at equilibrium. The demands obtained with a total market size equal to one leads to demands of the firms that are equivalent to their market shares, which are easier to interpret.

Firm $b$ does not restrict redemption. Finally, in scenario 3 (S3), both firms apply an unrestricted redemption policy. A firm imposes restriction of redemption if it only allows its customers to redeem their rewards after a pre-specified number of purchases. In the unrestricted LP, customers earn rewards with each purchase, which can be redeemed as early as the next purchase.

### 3.1. Customers choices

Similar to some of the previous research in the literature (e.g., Singh et al., 2008; Gandomi & Zolfaghari, 2013; Caminal & Matutes, 1990; Kim et al., 2001), we assume that customers' valuation for the product is sufficiently high that it exceeds the product prices. Therefore, customers buy one unit of the product in each period of purchasing from either Firm $a$ or Firm $b$. Customers are also allowed to switch between firms without penalty or extra cost. Consequently, with each purchase, a customer should decide among four alternatives: purchasing From Firm $a$ and not redeeming (denoted by A0), purchasing from Firm $a$ and redeeming (denoted by A1), purchasing from Firm $b$ and not redeeming (denoted by B0), and purchasing from Firm $b$ and redeeming (denoted by B1).

Customers are assumed rational and forward looking when making their purchasing and redemption decisions, so customer $j$ will choose the alternative $z \in \{$A0, A1, B0, B1$\}$ if his/her utility obtained from this alternative (denoted by $U_z^j$) is greater than the utility gained from the other choices. To represent random effects that can influence consumer utility, we consider that the customer $j$'s utility of choosing alternative $z$ is given by $U_z^j = D_z^j + \varepsilon_z^j$, where $D_z^j$ is the deterministic part representing consumer surplus and $\varepsilon_z^j$ is the random part. In line with the Multinomial Logit model (MNL) commonly used in the literature (e.g., Meissner & Strauss, 2012; Hahn, 2006; Perboli et al., 2014), the random parts are considered independently and identically Gumbel distributed (the type-1 extreme value), and the probability of choosing alternative $z$ by customer $j$ (denoted by $q_z^j$), as the one that results in the highest utility, is obtained via Equation (1).

$$q_z^j = \exp(D_z^j) \Big/ \Big(\exp\big(D_{A0}^j\big) + \exp\big(D_{A1}^j\big) + \exp\big(D_{B0}^j\big) + \exp\big(D_{B1}^j\big)\Big). \quad (1)$$

### 3.2. Valuation function

Referring to the mental accounting theory (Thaler, 1985), customers value cash and rewards differently. To model this mentality, we assume that consumers value $x$ amounts of reward less than they value the same amount of cash. Specifically, $x$ units of rewards is valued at $\alpha_v x$, where $\alpha_v$ is a positive parameter lower than 1 and represents consumers' unit valuation of rewards as compared to cash.

To model the goal-gradient theory in the consumer utility functions (e.g., Kivetz et al., 2006; Besanko & Winston, 1990), we consider that customers' evaluation of rewards is affected by the number of purchasing occasions they have left until they can redeem their points. In particular, the value of a point increases as the customer gets closer to the redemption time. Using a simple discounting formula (Crosson & Needles, 2008), one unit of reward is then valued by the customer at $1/(1+\alpha_d)^t$, where $\alpha_d$ is the interest rate for one period and varies in the range of (0, 1), and $t$ is the number of periods left until the customer redeems the reward.

Based on the above explanations, the value function ($V(x,t)$) representing consumers' evaluation of $x$ units of reward that can be redeemed after $t$ periods is as follows:

$$V(x,t) = \alpha_v x/(1+\alpha_d)^t. \quad (2)$$

### 3.3. Customers' deterministic surplus

In each alternative $z \in \{$A0, A1, B0, B1$\}$, we formulate customers' deterministic surplus obtained by their gains diminished by the losses they incur in such alternative. As is usually the case for restricted LPs offered as 'get one product free after $n$ purchases', we consider that customers of a firm offering a restricted LP can get a reward "in kind", i.e., a free product after completing the required number of purchasing imposed by the company. So when Firm $i$ ($i = \{a,b\}$) chooses a restricted redemption policy, it only allows its customers to redeem their rewards (a free product valued at $p_i$) after a pre-specified number of purchases, $N_i$, with $N_i$ an integer higher than 1.

In the unrestricted LP, customers are offered rewards at each purchase, which can be redeemed as early as the next purchase. In order to focus on the effects of restricting reward redemption, and exclude those of reward magnitude, we assume that customers are able to accumulate the unrestricted rewards they earn up to a maximum value equal to the price of one product. To do so, we assume the reward of Firm $i \in \{a, b\}$ to be a fraction of price $(p_i/M_i)$ if Firm $i$ does not restrict redemption, with $M_i$ an integer higher than 1. In summary, Firm $i \in \{a, b\}$ offers a free product (valued at $p_i$) as reward after $N_i$ purchases under the restricted redemption policy, and offers a reward valued at $(p_i/M_i)$ for each purchase under the unrestricted redemption policy. Note that the policy of not offering an LP is a special case of the unrestricted policy, where $N_i(M_i) = \infty, i = \{a, b\}$.

To track the customers' purchasing history, we denote $n_i^j$ as customer $j$'s number of purchases from Firm $i \in \{a, b\}$ after the last redemption. Consequently, one can say customer $j$ has to buy $(N_i - n_i^j)$ more times from Firm $i$ before he/she can redeem the reward of a free product if Firm $i$ restricts the redemption; while, this customer can immediately redeem $(n_i^j p_i/M_i)$ points of the firm that does not restrict redemption. For simplicity, we assume that with a restricted LP, customers are not offered a new reward if they are eligible to receive a free product $(0 \leq n_i^j \leq N_i)$. With the unrestricted LP, customers are not given a new reward if they have accumulated enough points to get a free product but have not redeemed their points $(0 \leq n_i^j \leq M_i)$.

Based on the above definitions, one can formulate customer $j$'s surplus in each alternative $z \in$\{A0, A1, B0, B1\}, $D_z^j$, for three different scenarios: both firms restrict redemption (S1), only Firm $a$ restricts redemptions (S2), and both firms do not restrict redemption (S3). Consequently, Firm $a$ restricts redemption in S1 and S2, while Firm $b$ restricts redemption only in S1.

Customer $j$'s deterministic surplus of purchasing from Firm $a$ and not redeeming ($D_{A0}^j$ ) consists in the value of purchasing the product $(v_a^j)$ diminished by the price $(p_a)$ in addition to the value of any accumulated points, which depends on whether the firm restricts rewards (S1 and S2) or not (S3). It is given by:

$$D_{A0}^{j} = \begin{cases} v_a^j - p_a & n_a^j = N_a \, in\ Scenario\ S1\ \&\ S2 \text{ or } n_a^j = M_a \, in\ Scenario\ S3 \\ v_a^j - p_a + V\left(p_a, N_a - n_a^j\right) & n_a^j < N_a, in\ Scenario\ S1\ \&\ S2 \\ v_a^j - p_a + V(p_a/M_a\,, 1) & n_a^j < M_a, in\ Scenario\ S3 \end{cases} \quad (3)$$

Customer $j$'s deterministic surplus of purchasing from Firm $a$ and redeeming ($D_{A1}^{j}$) consists in the value of purchasing the product diminished by the price ($v_a^j - p_a$) and by the value of the redeemed points, which depends on whether the firm restricts rewards (S1 and S2) or not (S3). If Firm $a$'s customer has not done the required number of purchases to receive a reward and has no points to redeem, his/her surplus would be equivalent to ($-\infty$). Therefore:

$$D_{A1}^{j} = \begin{cases} v_a^j - V(p_a, 0) & n_a^j = N_a, in\ Scenario\ S1\ \&\ S2 \\ v_a^j - p_a + n_a^j\, p_a/M_a - V\left(n_a^j\, p_a/M_a\,, 0\right) & n_a^j \neq 0, in\ Scenario\ S3 \\ -\infty & otherwise. \end{cases} \quad (4)$$

Similarly, $n_b^j$ denotes the number of customer $j$'s purchases from Firm $b$ after the last redemption, $p_b/M_b$ is the reward earned by purchasing from Firm $b$ (without redemption), and $N_b$ is the required number of purchases before earning a free product when Firm $b$ restricts redemption (in S1). Customer $j$'s surplus when purchasing from Firm $b$ and not redeeming ($D_{B0}^{j}$) consists in the value of purchasing the product ($v_b^j$) diminished by the price ($p_b$) in addition to the value of any accumulated points, which depends on whether the firm restricts rewards (S1) or not (S2 and S3). It is given by:

$$D_{B0}^{j} = \begin{cases} v_b^j - p_b & n_b^j = N_b\ in\ Scenario\ S1 \text{ or } n_b^j = M_b\ in\ Scenario\ S2\ \&\ S3 \\ v_b^j - p_b + V(p_b/M_b\,, 1) & n_b^j < M_b, in\ Scenario\ S2\ \&\ S3 \\ v_b^j - p_b + V\left(p_b, N_b - n_b^j\right) & n_b^j < N_b, in\ Scenario\ S1. \end{cases} \quad (5)$$

Finally, customer $j$'s surplus when purchasing from Firm $b$ and redeeming ($D_{B1}^{j}$) consists in the value of purchasing the product diminished by the price ($v_b^j - p_b$) and by the value of the redeemed points, which depends on whether the firm restricts rewards (S1) or not (S2 and S3). If Firm $b$'s customer has not done the required number of purchases to receive a reward and has no points to redeem, his/her surplus would be equivalent to ($-\infty$). This leads to the following formulation:

$$D_{B1}^{j} = \begin{cases} v_b^j - p_b + n_b^j p_b/M_b - V\left(n_b^j p_b/M_b\,, 0\right) & n_b^j \neq 0, in\ Scenario\ S2\ \&\ S3 \\ v_b^j - V(p_b, 0) & n_b^j = N_b, in\ Scenario\ S1 \\ -\infty & otherwise. \end{cases} \quad (6)$$

Given that a customer of a firm that does not restrict redemption chooses between redeeming all his/her points or none of them, it easy to show that a customer would get a higher surplus from redeeming all his/her stockpiled points rather than a part of them. This can be proved based on Equations (6) since, for any $x > 0$, $V(x, 0)$ is monotonously increasing in $x$. Finally, to simplify the analysis, and given that both firms sell similar products, we assume that all customers have similar preferences for the products of Firm $a$ and $b$, which leads to equal valuations ($v_b^j = v_a^j$ for any $j$).

### 3.4. The Firms' demand functions

Referring to Equations (1-6), the probability of choosing alternative $z$ by customer $j$ ($q_z^j$) is a function of the number of his/her previous purchases from Firm $i$ ($n_i^j$), with $i \in \{a, b\}$. Let $Q_z(n_a, n_b)$ be the probability of choosing alternative $z$ by any customer who has purchased $n_a$ and $n_b$ times from Firm $a$ and Firm $b$, respectively. In other words,

$$Q_z(n_a, n_b) = \left\{q_z^j \middle| \forall z \in \{A0, A1, B0, B1\}, n_a^j = n_a, n_b^j = n_b\right\}.$$

Therefore, the probability of a choice is equal for all customers who are characterized by $(n_a, n_b)$. We denote by $T(n_a, n_b)$ the set of such customers, then derive the flow chart of consumer choices. Figure 1 shows an example of such a flow chart for Scenario S2. The other scenarios' flow charts can be derived similarly to this one.

Based on this flow chart, denoting $\tau(n_a, n_b)$ as the number of customers in the set $T(n_a, n_b)$, one can derive $(N_a + 1) * (N_b + 1)$ independent equations for all values of $n_a$ and $n_b$. An illustrative example is included in Appendix B for clarity. Solving these equations simultaneously, we are able to obtain $\tau(n_a, n_b)$ for all values of $n_a$ and $n_b$.

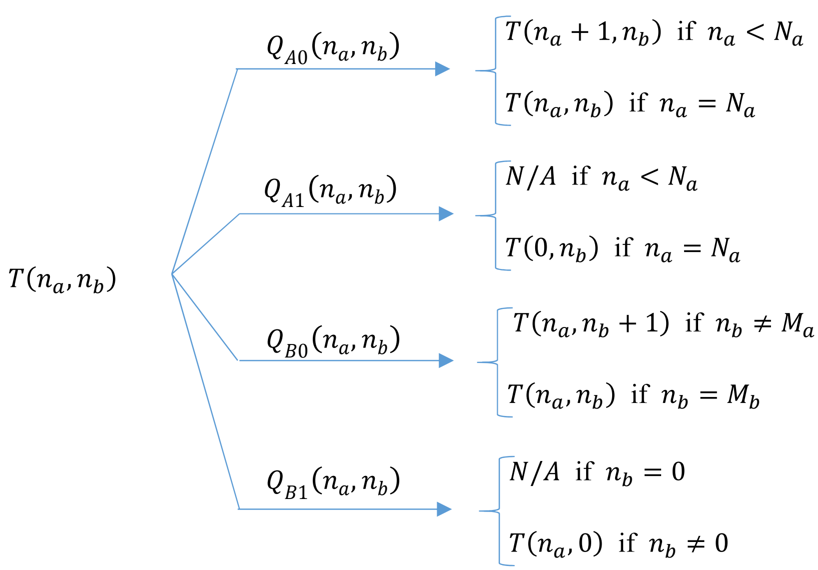



Figure 1: Flow chart of consumer choices in Scenario S2

### 3.5. The Firms' profit functions

Based on the previous sections, Firm $i$'s revenue ($R_i^S$) and cost of paid rewards ($C_i^S$) in scenario $S \in (S1, S2, S3)$ are given by:

$$\begin{cases} R_a^{S1} = p_a \sum_{n_a=0}^{N_a} \sum_{n_b=0}^{N_b} \tau(n_a, n_b) \left(Q_{A0}(n_a, n_b) + Q_{A1}(n_a, n_b)\right) \\ C_a^{S1} = p_a \sum_{n_a=0}^{N_a} \sum_{n_b=0}^{N_b} \tau(n_a, n_b)\, Q_{A1}(n_a, n_b) \end{cases} \tag{7}$$

$$\begin{cases} R_a^{S2} = p_a \sum_{n_a=0}^{N_a} \sum_{n_b=0}^{M_b} \tau(n_a, n_b) \left(Q_{A0}(n_a, n_b) + Q_{A1}(n_a, n_b)\right) \\ C_a^{S2} = p_a \sum_{n_a=0}^{N_a} \sum_{n_b=0}^{M_b} \tau(n_a, n_b)\, Q_{A1}(n_a, n_b) \end{cases} \tag{8}$$

$$\begin{cases} R_a^{S3} = p_a \sum_{n_a=0}^{M_a} \sum_{n_b=0}^{M_b} \tau(n_a, n_b) \left(Q_{A0}(n_a, n_b) + Q_{A1}(n_a, n_b)\right) \\ C_a^{S3} = \sum_{n_a=0}^{M_a} \sum_{n_b=0}^{M_b} \tau(n_a, n_b)\, Q_{A1}(n_a, n_b) n_a\, p_a / M_a \end{cases} \tag{9}$$

$$\begin{cases} R_b^{S1} = p_b \sum_{n_a=0}^{N_a} \sum_{n_b=0}^{N_b} \tau(n_a, n_b) \left(Q_{B0}(n_a, n_b) + Q_{B1}(n_a, n_b)\right) \\ C_b^{S1} = p_b \sum_{n_a=0}^{N_a} \sum_{n_b=0}^{N_b} \tau(n_a, n_b)\, Q_{B1}(n_a, n_b) \end{cases} \tag{10}$$

$$\begin{cases} R_b^{S2} = p_b \sum_{n_a=0}^{N_a} \sum_{n_b=0}^{M_b} \tau(n_a, n_b)\left(Q_{B0}(n_a, n_b) + Q_{B1}(n_a, n_b)\right) \\ C_b^{S2} = \sum_{n_a=0}^{N_a} \sum_{n_b=0}^{M_b} \tau(n_a, n_b)\, Q_{B1}(n_a, n_b) n_b\, p_b / M_b \end{cases} \tag{11}$$

$$\begin{cases} R_b^{S3} = p_b \sum_{n_a=0}^{M_a} \sum_{n_b=0}^{M_b} \tau(n_a, n_b)\big(Q_{B0}(n_a, n_b) + Q_{B1}(n_a, n_b)\big) \\ C_b^{S3} = \sum_{n_a=0}^{M_a} \sum_{n_b=0}^{M_b} \tau(n_a, n_b)\, Q_{B1}(n_a, n_b) n_b\, p_b / M_b \end{cases} \tag{12}$$

Each firm's profit is given by its expected revenue from selling the product minus its expected cost of rewards redemption. Therefore Firm $i$'s profit in scenario $S$, denoted by $I_a^S$, is given by:

$$I_i^S = R_i^S - C_i^S. \tag{13}$$

## 4. Solving the model

In each scenario (S1, S2 and S3), Firms $a$ and $b$ play a Nash game, i.e., each firm sets its decision variables that maximize its profit and both firms make their decisions simultaneously without knowing each other's decisions. Comparison of the firms' profits in each scenario (sub-game) will then lead to identifying the equilibrium solution for the game played by Firm $a$ and Firm $b$ in which they decide about restricting or not restricting their redemption policies.

Table 1: The firms' decision variables in different scenarios

| Scenario | Description | Firm $a$'s decision variables | Firm $b$'s decision variables |
|---|---|---|---|
| S1 | Both firms restrict reward redemption | $N_a$ and $p_a$ | $N_b$ and $p_b$ |
| S2 | One firm restricts reward redemption | $N_a$ and $p_a$ | $M_b$ and $p_b$ |
| S3 | Neither firm restricts reward redemption | $M_a$ and $p_a$ | $M_b$ and $p_b$ |

In each scenario, each firm decides of its price while the loyalty program decision variables vary by scenario. In the first scenario (S1), each Firm $i \in \{a, b\}$ restricts redemption and decides of the required number of purchasing for a customer to qualify for a reward ($N_i$). In the second scenario (S2), only Firm $a$ restricts redemption and decides of ($N_a$), while Firm B does not restrict redemption and sets ($M_b$). In the third scenario (S3), both firms do not restrict redemption and each firm decides about its reward variables ($M_i$). Table 1 summarizes the firms' decision variables in the different scenarios.

As can be derived from equations (1-13), the firms' profits are functions of their decision variables, $N(M)_a$, $p_a$, $N(M)_b$ and $p_b$, and of the behavioural parameters ($\alpha_v$ and $\alpha_d$). In each scenario $S \in (S1, S2, S3)$, we solve for the firms' optimal decisions by simultaneously maximizing the firms' profits. For example, we solve the following problems for Scenario S2:

(I) $max_{N_a,p_a} I_a^{S2}$

s.t. $N_a$ is an integer, $p_a > 0$.

(II) $max_{M_b,p_b} I_b^{S2}$

s.t. $M_b$ is an integer, $p_b > 0$.

An analytical solution of these problems, if available, is difficult to derive because of the high nonlinearity of the profit functions (Equation 13). Therefore, we use an iterative algorithm to get the optimal solutions. This algorithm is described in Table 2. In each scenario, it numerically seeks the Nash equilibrium, where neither firm can increase its profit by unilaterally deviating to any other possible solution, and finds the closest condition to the equilibrium with an error of (0.001). Not converging in this algorithm is interpreted as either having more than one equilibrium or no equilibrium.

Next, we set the range of numerical values for the model's variables and parameters. In restricted LPs of the type "buy $n$ times, get one free", $n$ is usually limited to no more than 10 purchases (e.g., McDonald, Starbucks and Second Cup loyalty cards). Therefore, we vary the positive integer variable $N_i$ between 1 and 10. In unrestricted LPs, customers of Firm $i$ get a reward equal to $1/M_i$. We then vary the positive integer variable $M_i$ between 1 and 100. Finally, we do not set a maximum limit for the pricing variables, i.e., and for each combination of ($N_a(M_a)$, $N_b(M_b)$, $p_{b(a)}$), an optimal positive price ($p_{a(b)}$) is found which maximizes Firm $a(b)$'s profit. Employing Matlab's fmincon function, we apply an interior-point algorithm based on Byrd et al. (2000) to reach the optimal price with the accuracy of 32 decimal digits.

Table 2: Equilibrium finder algorithm

| | task |
|---|---|
| 1 | Find optimal response $[N_b(M_b)^1, p_b{}^1]$ to initial strategy of $[N_a(M_a)^1 = 1, p_a{}^1 = 0]$ |
| 2 | Find optimal response $[N_a(M_a)^2, p_a{}^2]$ to the strategy of $[N_b(M_b)^1, p_b{}^1]$ |
| 3 | Find optimal response $[N_b(M_b)^2, p_b{}^2]$ to the strategy of $[N_a(M_a)^2, p_a{}^2]$ |
| 4 | $x = abs([N_a(M_a)^2, p_a{}^2, N_b(M_b)^2, p_b{}^2] - [N_a(M_a)^1, p_a{}^1, N_b(M_b)^1, p_b{}^1])$ |
| 5 | **While** $x \geq converge\ threshold$, **do** |
| 6 | $[N_b(M_b)^1, p_b{}^1] = [N_b(M_b)^2, p_b{}^2]$ |
| 7 | Repeat step 2 – step 4 |
| 8 | **end while** |

In our framework, parameters $\alpha_v$ and $\alpha_d$ are bounded in the (0, 1) interval. We consider the range of [0.1, 0.9] with a step size of 0.1 for parameter $\alpha_v$ and a smaller range of [0.1, 0.5] with the same step size for parameter $\alpha_d$. Note that $\alpha_d = 0.5$ means that a reward loses half of its value after one single purchasing period, a higher depreciation of the reward would not be a realistic assumption. This results in 45 different combinations of the behavioral parameters. Next, we solve for the optimal solutions in the three scenarios. Then, we identify the Nash equilibrium for the general game in which the firms decide about restricting or not restricting their redemption policies.

## 5. Results and discussion

Applying the numerical method explained in the previous section, we obtain each firm's optimal decision variables and profits for all 45 combinations of $\alpha_v$ (reward valuation coefficient) and $\alpha_d$ (time valuation coefficient) in each scenario. We present the solutions obtained in each scenario (S1, S2, and S3), then we compare these results.

### 5.1. Scenario S1

In S1, both firms restrict redemption. They decide of the required number of purchases a customer needs to be rewarded a free product and the price of the product to maximize their profits. Using the algorithm explained in Table 2, we numerically obtain a Nash equilibrium ($N_i$ and $p_i, i = a, b$) for each combination of $\alpha_v$ and $\alpha_d$. As expected, the firms' optimal decision variables and thereby optimal profits are equal in S1. They are shown in Table 3 for different combinations of $\alpha_v$ and $\alpha_d$.

Table 3: Optimal decision variables and profit in Scenario S1 for combinations of $\alpha_v$ & $\alpha_d$

| $\alpha_v$ | $\alpha_d$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.1 | | | 0.2 | | | 0.3 | | | 0.4 | | | 0.5 | | |
| | $N_i$ | $p_i$ | $I_i$ | $N_i$ | $p_i$ | $I_i$ | $N_i$ | $p_i$ | $I_i$ | $N_i$ | $p_i$ | $I_i$ | $N_i$ | $p_i$ | $I_i$ |
| 0.1 | 10 | 2.127 | 0.968 | 10 | 2.083 | 0.948 | 10 | 2.059 | 0.937 | 10 | 2.044 | 0.931 | 10 | 2.035 | 0.926 |
| 0.2 | 9 | 2.287 | 1.031 | 7 | 2.219 | 0.974 | 8 | 2.146 | 0.956 | 10 | 2.090 | 0.952 | 10 | 2.071 | 0.943 |
| 0.3 | 6 | 2.547 | 1.096 | 5 | 2.419 | 1.014 | 5 | 2.319 | 0.972 | 5 | 2.253 | 0.945 | 5 | 2.207 | 0.926 |
| 0.4 | 5 | 2.847 | 1.193 | 4 | 2.661 | 1.075 | 4 | 2.509 | 1.014 | 4 | 2.407 | 0.974 | 4 | 2.336 | 0.945 |
| 0.5 | 5 | 3.140 | 1.317 | 4 | 2.865 | 1.159 | 3 | 2.765 | 1.057 | 3 | 2.624 | 1.005 | 3 | 2.522 | 0.966 |
| 0.6 | 5 | 3.399 | 1.428 | 4 | 3.043 | 1.234 | 3 | 2.905 | 1.116 | 3 | 2.737 | 1.053 | 3 | 2.616 | 1.007 |
| 0.7 | 5 | 3.483 | 1.468 | 4 | 3.125 | 1.273 | 3 | 2.959 | 1.144 | 3 | 2.796 | 1.083 | 3 | 2.672 | 1.036 |
| 0.8 | 6 | 3.368 | 1.460 | 4 | 3.063 | 1.255 | 4 | 2.864 | 1.175 | 3 | 2.776 | 1.084 | 3 | 2.672 | 1.045 |
| 0.9 | 7 | 3.184 | 1.410 | 5 | 2.928 | 1.247 | 4 | 2.781 | 1.149 | 3 | 2.683 | 1.060 | 3 | 2.615 | 1.033 |

As can be derived from Table 3, the optimal price and the optimal profit increase when the customers' time valuation parameter ($\alpha_d$) decreases. The optimal number of required purchases to get reward ($N_i$) also decreases with $\alpha_d$ for high enough values of the reward valuation parameter ($\alpha_v$). In other words, the firms should offer a higher price and reward earlier when their customers highly value time, and this condition results in higher profits when both firms restrict redemption. **Note also that Table 3 shows an optimal $N_i$ equal to 10 for very low values of $\alpha_v$. Since 10 is the upper bound value set in our numerical analysis, we cannot be sure about the real optimal value of this variable in these conditions.**

Furthermore, we observe that for each $\alpha_d$, the optimal price and profit increase with higher values of $\alpha_v$ then decrease once $\alpha_v$ exceeds a threshold value. Table 3 also reveals that for low values of $\alpha_d$, there is a trade-off area for $\alpha_v$ under which the required number of purchases for rewarding is minimum.

However, for low values of $\alpha_d$, the required number of purchases for rewarding increases with higher levels of reward valuation ($\alpha_v$).

### 5.2. Scenario S2

In Scenario S2, Firm $a$ restricts redemption and decides of its price ($p_a$) and of the required number of purchases to receive the reward ($N_a$). Firm $b$ does not restrict redemption and sets its price ($p_b$) and reward ($M_b$). Using our numerical algorithm (Table 2), we get the Nash equilibrium for the game between Firm $a$ and $b$, where the optimal decision of each firm is the best response to the other firm's strategy. Tables 4 and 5 show Firm $a$'s and $b$'s optimal decision variables and profits, respectively in Scenario S2 for the different combinations of $\alpha_v$ and $\alpha_d$.

Table 4: Firm $a$'s optimal decision variables and profits in Scenario S2 for combinations of $\alpha_v$ & $\alpha_d$

| $\alpha_v$ | $\alpha_d$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.1 | | | 0.2 | | | 0.3 | | | 0.4 | | | 0.5 | | |
| | $N_a$ | $p_a$ | $I_a$ | $N_a$ | $p_a$ | $I_a$ | $N_a$ | $p_a$ | $I_a$ | $N_a$ | $p_a$ | $I_a$ | $N_a$ | $p_a$ | $I_a$ |
| 0.1 | 10 | 2.022 | 0.873 | 10 | 1.980 | 0.855 | 10 | 1.958 | 0.845 | 10 | 1.944 | 0.840 | 10 | 1.935 | 0.836 |
| 0.2 | 8 | 2.195 | 0.930 | 7 | 2.123 | 0.889 | 8 | 2.049 | 0.869 | 9 | 2.002 | 0.858 | 10 | 1.972 | 0.852 |
| 0.3 | 6 | 2.437 | 1.001 | 5 | 2.327 | 0.937 | 5 | 2.233 | 0.900 | 5 | 2.171 | 0.876 | 5 | 2.128 | 0.859 |
| 0.4 | 5 | 2.726 | 1.092 | 4 | 2.567 | 0.999 | 4 | 2.425 | 0.947 | 4 | 2.331 | 0.912 | 4 | 2.264 | 0.887 |
| 0.5 | 5 | 2.998 | 1.200 | 4 | 2.759 | 1.074 | 3 | 2.686 | 0.998 | 3 | 2.554 | 0.952 | 3 | 2.459 | 0.919 |
| 0.6 | 5 | 3.252 | 1.306 | 4 | 2.932 | 1.145 | 3 | 2.822 | 1.053 | 3 | 2.665 | 0.998 | 3 | 2.550 | 0.957 |
| 0.7 | 5 | 3.380 | 1.378 | 4 | 3.034 | 1.198 | 3 | 2.891 | 1.092 | 3 | 2.732 | 1.033 | 3 | 2.611 | 0.988 |
| 0.8 | 6 | 3.320 | 1.415 | 4 | 3.019 | 1.217 | 4 | 2.803 | 1.123 | 3 | 2.734 | 1.051 | 3 | 2.625 | 1.007 |
| 0.9 | 7 | 3.186 | 1.412 | 5 | 2.908 | 1.228 | 4 | 2.756 | 1.127 | 3 | 2.671 | 1.050 | 3 | 2.589 | 1.013 |

Table 4 shows almost the same trend for the optimal reward timing and optimal price of the firm that restricts redemption in Scenario S2 as those in Scenario S1. However, Firm $a$'s optimal profit indicates an increasing trend when reward valuation ($\alpha_v$) increases and time distance value ($\alpha_d$) decreases.

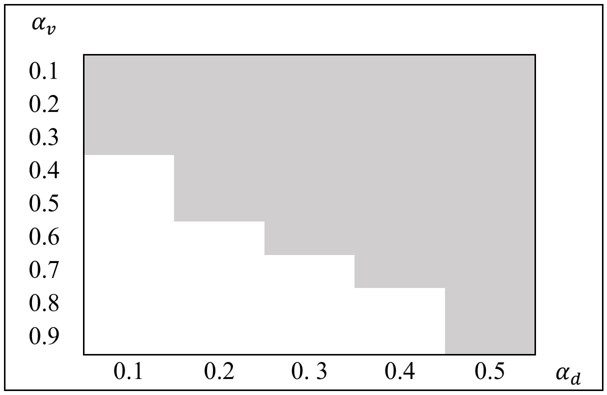
Table 5: Firm $b$'s optimal decision variables and profits in Scenario S2 for combinations of $\alpha_v$ & $\alpha_d$

| $\alpha_v$ | $\alpha_d$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.1 | | | 0.2 | | | 0.3 | | | 0.4 | | | 0.5 | | |
| | $1/M_b$ | $p_b$ | $I_b$ | $1/M_b$ | $p_b$ | $I_b$ | $1/M_b$ | $p_b$ | $I_b$ | $1/M_b$ | $p_b$ | $I_b$ | $1/M_b$ | $p_b$ | $I_b$ |
| 0.1 | 0.01 | 2.117 | 1.106 | 0.01 | 2.116 | 1.105 | 0.01 | 2.116 | 1.105 | 0.01 | 2.115 | 1.105 | 0.01 | 2.115 | 1.104 |
| 0.2 | 0.01 | 2.112 | 1.100 | 0.01 | 2.105 | 1.093 | 0.01 | 2.109 | 1.098 | 0.01 | 2.113 | 1.101 | 0.01 | 2.116 | 1.105 |
| 0.3 | 0.01 | 2.105 | 1.093 | 0.01 | 2.092 | 1.081 | 0.01 | 2.090 | 1.078 | 0.01 | 2.088 | 1.077 | 0.01 | 2.087 | 1.076 |
| 0.4 | 0.01 | 2.103 | 1.091 | 0.01 | 2.086 | 1.074 | 0.01 | 2.081 | 1.070 | 0.01 | 2.078 | 1.067 | 0.01 | 2.076 | 1.065 |
| 0.5 | 0.01 | 2.108 | 1.095 | 0.01 | 2.090 | 1.078 | 0.01 | 2.070 | 1.058 | 0.01 | 2.065 | 1.054 | 0.01 | 2.062 | 1.050 |
| 0.6 | 0.01 | 2.105 | 1.092 | 0.01 | 2.089 | 1.077 | 0.01 | 2.071 | 1.058 | 0.01 | 2.066 | 1.054 | 0.01 | 2.063 | 1.051 |
| 0.7 | 0.01 | 2.080 | 1.067 | 0.01 | 2.076 | 1.063 | 0.01 | 2.060 | 1.047 | 0.01 | 2.060 | 1.048 | 0.01 | 2.059 | 1.047 |
| 0.8 | 0.04 | 2.095 | 1.034 | 0.01 | 2.046 | 1.033 | 0.01 | 2.060 | 1.047 | 0.01 | 2.045 | 1.032 | 0.01 | 2.049 | 1.037 |
| 0.9 | 0.20 | 2.304 | 0.997 | 0.01 | 2.030 | 1.016 | 0.01 | 2.034 | 1.021 | 0.01 | 2.021 | 1.009 | 0.01 | 2.032 | 1.020 |

On the other hand, Table 5 reveals that Firm $b$'s optimal price and profit increase when customers' reward valuation decreases. The only exception to this trend is when customers have very low valuation of time distance. Table 5 also shows an optimal reward percentage ($1/M_b$) equal to $0.01$ for most values of $\alpha_v$ and $\alpha_d$. This means that the optimal reward value for $M_b$ is found equal to the upper bound value set in our numerical analysis (100) for most values of $\alpha_v$ and $\alpha_d$. Therefore, we cannot be sure about the real optimal value of this variable in these conditions since it can be any number greater than 100 that our numerical method does not catch.

Comparison of Firm $a$'s and Firm $b$'s profits in S2 gives the results showcased in Table 6. The shaded area indicates values of $\alpha_v$ and $\alpha_d$ where Firm $b$ earns higher profit than Firm $a$. This means that in this area, the firm that does not restrict reward redemption earns higher profit than the competing firm that restricts redemption if the required number of purchases to get reward is equal to or greater than 10. In the remaining unshaded area of Table 6, the restricted redemption policy is more profitable than the unrestricted one if the percentage reward is equal or greater than 1% of the price.

Table 6: Comparison of firms' profits in S2
(Firm $b$ earns higher profit than Firm $a$ in the shaded area)




### 5.3. Scenario S3

In Scenario S3, both firms do not restrict redemption. Therefore, each firm decides about its price and reward. Since Scenario S3 is a symmetric condition, as expected, the optimal decisions variables are equal for both firms.

Table 7: Optimal decision variables and profit in Scenario S3 for combinations of $\alpha_v$ & $\alpha_d$

| $\alpha_v$ | $\alpha_d$ | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | 0.1 | | | 0.2 | | | 0.3 | | | 0.4 | | | 0.5 | | |
| | $1/M_i$ | $p_i$ | $I_i$ | $1/M_i$ | $p_i$ | $I_i$ | $1/M_i$ | $p_i$ | $I_i$ | $1/M_i$ | $p_i$ | $I_i$ | $1/M_i$ | $p_i$ | $I_i$ |
| 0.1 | 0.01 | 2.006 | 0.996 | 0.01 | 2.006 | 0.996 | 0.01 | 2.006 | 0.996 | 0.01 | 2.006 | 0.996 | 0.01 | 2.006 | 0.996 |
| 0.2 | 0.01 | 2.007 | 0.997 | 0.01 | 2.007 | 0.997 | 0.01 | 2.007 | 0.997 | 0.01 | 2.006 | 0.997 | 0.01 | 2.006 | 0.996 |
| 0.3 | 0.01 | 2.008 | 0.997 | 0.01 | 2.008 | 0.997 | 0.01 | 2.007 | 0.997 | 0.01 | 2.007 | 0.997 | 0.01 | 2.007 | 0.997 |
| 0.4 | 0.01 | 2.009 | 0.998 | 0.01 | 2.008 | 0.997 | 0.01 | 2.008 | 0.997 | 0.01 | 2.008 | 0.997 | 0.01 | 2.007 | 0.997 |
| 0.5 | 0.01 | 2.010 | 0.998 | 0.01 | 2.009 | 0.998 | 0.01 | 2.009 | 0.998 | 0.01 | 2.008 | 0.997 | 0.01 | 2.008 | 0.997 |
| 0.6 | 0.01 | 2.011 | 0.999 | 0.01 | 2.010 | 0.998 | 0.01 | 2.009 | 0.998 | 0.01 | 2.009 | 0.998 | 0.01 | 2.008 | 0.997 |
| 0.7 | 0.01 | 2.011 | 0.999 | 0.01 | 2.011 | 0.999 | 0.01 | 2.010 | 0.998 | 0.01 | 2.009 | 0.998 | 0.01 | 2.009 | 0.998 |
| 0.8 | 0.05 | 2.062 | 0.999 | 0.01 | 2.011 | 0.999 | 0.01 | 2.010 | 0.999 | 0.01 | 2.010 | 0.998 | 0.01 | 2.009 | 0.998 |
| 0.9 | 0.20 | 2.304 | 0.997 | 0.01 | 2.012 | 0.999 | 0.01 | 2.011 | 0.999 | 0.01 | 2.010 | 0.998 | 0.01 | 2.010 | 0.998 |

Table 7 deals with the optimal decision variables and profit in Scenario S3. It shows that, in Scenario S3, firms should increase their prices when customers' time valuation decreases and reward valuation increases. This trend is also observed for the firms' optimal profit except when customers highly value rewards but have a low valuation for time distance ($\alpha_v = 0.9$ and $\alpha_d = 0.1$). Under this condition, profits are lower for lower values of $\alpha_v$. This is mainly due to the higher value of reward percentage offered in this case.

### 5.4. Solving the general game

To find the optimal strategy for each firm given different LP choices by the competitor (restricted vs. unrestricted redemption policy), we now compare the solutions obtained in the three scenarios of S1, S2, and S3. We start by comparing the optimal prices and find that Firm $b$ prices are higher in S2 than in S3. This means that a firm that does not restrict redemption should offer a higher price if its competitor restricts redemption in comparison with the condition where its competitor does not restrict redemption. Similar comparisons between optimal prices in Scenario S1 and those of Firm $a$ in Scenario S2 show that a firm that applies restricted redemption policy should offer a higher price if its competitor also restricts redemption.

Knowing the firms' profits in each of the scenarios, we can now solve the general game in which each firm (player) decides about setting or not setting reward restriction given the other firm's optimal reactions. The firms' payoff matrix of this game for each combination of $\alpha_v$ and $\alpha_d$ can be derived as follows.

| Firm $b$ / Firm $a$ | Restricting | Unrestricting |
|---|---|---|
| Restricting | $(I_a^{S1}, I_b^{S1})$ | $(I_a^{S2}, I_b^{S2})$ |
| Unrestricting | $(I_b^{S2}, I_a^{S2})$ | $(I_a^{S3}, I_b^{S3})$ |

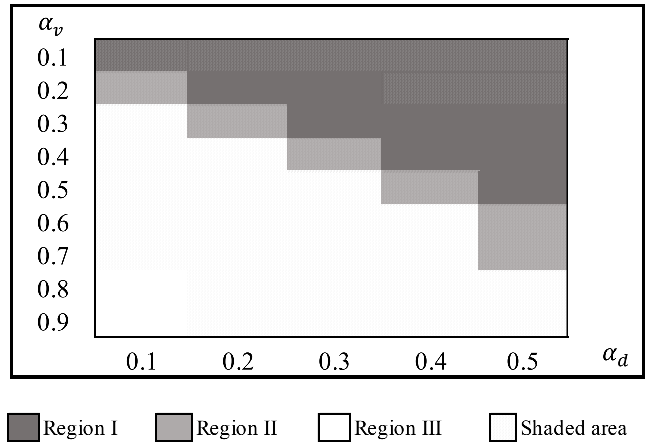
By comparing the obtained profits in scenarios S1, S2 and S3 (Tables 3, 4, 5 and 7) for each of the 45 combinations of parameters $\alpha_v$ and $\alpha_d$, one can categorize the results in different regions as shown in Table 8 (see Table A.2 in Appendix for a scaled version of Table 8).

In Region I of this table, an unrestricted redemption policy by both firms (Scenario S3) is the dominant Nash equilibrium, meaning that both firms should allow their customers to redeem their earned rewards at the next purchasing occasion. Note that this region is characterized by low reward valuation ($\alpha_v$), with lower values of time distance valuation ($\alpha_d$) required for the lowest levels of $\alpha_v$. S3 is the Nash equilibrium in this region for the following two reasons. First, each firm earns a higher profit with an unrestricted reward policy in the asymmetric games. Second, both firms earn higher profits when they both do not restrict redemption than when they both do (i.e., in Scenario S1). Comparisons of optimal strategies across scenarios S1 and S3 show that although firms charge higher prices in S3 than in S1, they also pay lower rewards to consumers. Overall, the lower cost of rewards results in higher profits in S3 than in S1 (Tables A.3-A.6 show the firms' revenue and cost in Scenario S1 and S3).

Region II in Table 8 is characterized by mid-range values of both parameters $\alpha_v$ and $\alpha_d$. In this region, the competing firms face a Prisoner dilemma which results in both firms choosing an unrestricted redemption policy at equilibrium (Scenario S3). This is despite the fact that both firms would earn higher profits when they both restrict redemption. The Prisoner dilemma situation arises because each firm earns a higher profit by unilaterally restricting redemption while the competitor does not. In this region of the parameters, the Nash equilibrium in S3 generates lower income and reward costs than in S1 with a restricted redemption by both firms, leading to overall lower profits than in S1.

In Region III of Table 8, restricting redemption by both firms (S1) is the dominant Nash equilibrium. This area is characterized by higher values of both reward and time distance parameters. In this region, the firms' optimal profits in Scenario S1 are higher than in S3, mainly because they both earn higher income levels in S1 and the gains in income exceed the additional incurred cost of rewards for both firms. Further, in the asymmetric games, each firm can increase its profit by restricting redemption while the competitor does not.

Table 8: Different regions of the game between Firm $a$ and Firm $b$




**Finally, as mentioned before, for some values of $\alpha_v$ and $\alpha_d$, the true optimal values of the reward decisions $N_i$ and $M_i$ cannot be identified with certainty due to the derivation of the numerical solution (the upper bound for $N_i$ and $M_i$ are restricted to 10 and 100, respectively). Therefore, the equilibrium for the general game also cannot be identified with certainty when a firm whose reward decision variable is equal to the bound value (i.e., $N_i = 10$ or $M_i = 100$) gains less profit than its competitor. This region is represented by the shaded area in Table 8.**

## 6. Conclusion

This paper investigates whether it is profitable for competing firms to restrict redemption of their loyalty rewards. LPs with unrestricted redemption policy provide rewards in form of a price discount to their customers that can be redeemed as soon as the next purchase. However, LPs with restricted redemption forces customers to accumulate rewards and to delay redemption at a specified occasion. In this paper, reward restriction is studied in the form of the commonly used “buy *n* times, get one free” loyalty program cards. These kinds of LPs are commonly used in a variety of industries characterized by uniform pricing, e.g., coffee shops, where LPs can help increase the possibility of repeat customer purchase. The motivation for this research is the lack of clear guidance in the literature about the impact of such restrictions and the observation in practice of different restricted and unrestricted LP programs offered by competing firms.

We use a game theoretic approach to explain why and when adopting the restricted redemption policy can be beneficial for competing firms given different customers' valuations of time and rewards. To the best of our knowledge, this is the first attempt to study the strategic impact of restricting or not restricting the redemption policies of loyalty programs in a competitive environment. To address this research problem, we developed and solved a game theoretic model of a market served by two firms. Using a numerical analysis, we obtain Nash equilibrium solutions for three scenarios (games): two symmetric games where, (1) both firms restrict reward redemption, and (2) neither firm restricts reward redemption; and (3) one asymmetric game where only one firm restricts reward redemption. In each of these three games, the firm that restricts redemption determines its price and the required number of purchasing before a customer is rewarded a free product. The firm that applies unrestricted redemption sets regular price and its reward as percentage of its price. Comparing optimal profits across scenarios, an equilibrium solution is found for the main game in which each firm decides whether or not to restrict the redemption, given the competitor's reaction.

Our main findings indicate that each firm's optimal strategies are significantly affected by whether the competitor decides to restrict or not to restrict redemption. For example, a firm that restricts reward redemption should offer a higher price if its competitor also restricts redemption.

Further, the dominant strategy of the game depends on customers' valuations of time and rewards. In particular, both firms should not restrict their reward redemption, i.e., should allow their customers to redeem their earned rewards at the next purchasing occasion when customers do not highly value rewards and are willing to wait to redeem their earned rewards (i.e., have low time valuation). Under such conditions, the competing firms gain a lower income but pay lower rewards to consumers with unrestricted reward policy than with a restricted one, which ultimately results in higher profits when reward redemption restrictions are lifted.

We also find that unrestricted redemption is still the Nash equilibrium when customer valuation of time and rewards are slightly higher. However, in this case, the competing firms face a Prisoner dilemma situation with each firm earning a higher profit by restricting redemption while the competitor does not. This is because when both firms restrict redemption, they earn lower income levels and incur lower reward costs than when they both restrict redemption, leading to overall lower profits. Finally, when customers highly value reward but not time, both firms should restrict redemption at equilibrium, mainly because they both earn higher income levels and the gains in income exceed the additional incurred cost of rewards.

This research provides a preliminary understanding of the effects of some customers' behavioural factors on the design of loyalty programs, in particular restricting the redemption policy. The analysis could be extended in several ways to address issues overlooked in this paper. For instance, our model is useful for studying the market of an undifferentiated product. Future research can adapt our model to study other set-ups such as asymmetric products where customers might have significantly different evaluations for each firm's product. **This can change our results since the firms' market shares in scenarios where both firms either offer or not offer an LP will be asymmetric and therefore their optimal choices to whether implement or not an LP might be affected by the consumer differing preferences for their products. Indeed, some empirical studies have showed that LPs mainly benefit large-share brands and those firms with a previously established competitive advantage (Sharp & Sharp, 1997; Meyer-Waarden & Benavent, 2007; Leenheer et al., 2007). Another extension of this study would be to consider a dynamic model where the firms' decisions (e.g., price) depend on time. Such situations are particularly relevant for LPs in the hoteling or airline industry for example where prices are changed regularly. A dynamic formulation can also capture other consumer behavior over time such as inertia or current market trends.**

## Appendix A

Table A.1: List of notations

| | Definition |
|---|---|
| $\alpha_v$ | Customers' reward evaluation coefficient |
| $\alpha_d$ | Customers' time distance evaluation coefficient |
| $N_i$ | Required number of purchases to receive Firm $i$'s reward when Firm $i$ restricts the redemption |
| $1/M_i$ | Fraction of price Firm $i$ gives as reward point when Firm $i$ does not restricts the redemption |
| $p_i$ | Firm $i$'s price |
| $n_i$ | Number of purchases from Firm $i$ after the last redemption |
| $R_i^S$ | Firm $i$'s revenue in scenario S |
| $C_i^S$ | Firm $i$'s cost (paid reward) in scenario S |
| $I_i^S$ | Firm $i$'s profit in scenario S |
| $U_z^j$ | Customer $j$'s utility of choosing alternative $z$ |
| $D_z^j$ | Deterministic part of customer $j$'s utility of choosing alternative $z$ |
| $\varepsilon_z^j$ | Random part of customer $j$'s utility of choosing alternative $z$ |
| $q_z^j$ | Probability of choosing alternative $z$ by customer $j$ |
| $\tau(n_a, n_b)$ | Number of customers whose variables are $(n_a, n_b)$ |
| $Q_z(n_a, n_b)$ | Probability of choosing alternative z by the customers whose variables are $(n_a, n_b)$ |

Table A.2: Scaled payoff matrices of the game between "restricted LP" and "unrestricted LP" policies for combinations of $\alpha_v$ & $\alpha_d$

| $\alpha_v$ \ $\alpha_d$ | 0.1 | | 0.2 | | 0.3 | | 0.4 | | 0.5 | |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.1 | 2,2 | 1,4 | 2,2 | 1,4 | 2,2 | 1,4 | 2,2 | 1,4 | 2,2 | 1,4 |
| | 4,1 | 3,3 | 4,1 | 3,3 | 4,1 | 3,3 | 4,1 | 3,3 | 4,1 | 3,3 |
| 0.2 | 3,3 | 1,4 | 2,2 | 1,4 | 2,2 | 1,4 | 2,2 | 1,4 | 2,2 | 1,4 |
| | 4,1 | 2,2 | 4,1 | 3,3 | 4,1 | 3,3 | 4,1 | 3,3 | 4,1 | 3,3 |
| 0.3 | 4,4 | 2,3 | 3,3 | 1,4 | 2,2 | 1,4 | 2,2 | 1,4 | 2,2 | 1,4 |
| | 3,2 | 1,1 | 4,1 | 2,2 | 4,1 | 3,3 | 4,1 | 3,3 | 4,1 | 3,3 |
| 0.4 | 4,4 | 3,2 | 4,4 | 2,3 | 3,3 | 1,4 | 2,2 | 1,4 | 2,2 | 1,4 |
| | 2,3 | 1,1 | 3,2 | 1,1 | 4,1 | 2,2 | 4,1 | 3,3 | 4,1 | 3,3 |
| 0.5 | 4,4 | 3,2 | 4,4 | 2,3 | 3,3 | 2,4 | 3,3 | 1,4 | 2,2 | 1,4 |
| | 2,3 | 1,1 | 3,2 | 1,1 | 4,2 | 1,1 | 4,1 | 2,2 | 4,1 | 3,3 |
| 0.6 | 4,4 | 3,2 | 4,4 | 3,2 | 4,4 | 2,3 | 3,3 | 2,4 | 3,3 | 1,4 |
| | 2,3 | 1,1 | 2,3 | 1,1 | 3,2 | 1,1 | 4,2 | 1,1 | 4,1 | 2,2 |
| 0.7 | 4,4 | 3,2 | 4,4 | 3,2 | 4,4 | 3,2 | 4,4 | 2,3 | 3,3 | 1,4 |
| | 2,3 | 1,1 | 2,3 | 1,1 | 2,3 | 1,1 | 3,2 | 1,1 | 4,1 | 2,2 |
| 0.8 | 4,4 | 3,2 | 4,4 | 3,2 | 4,4 | 3,2 | 4,4 | 3,2 | 4,4 | 2,3 |
| | 2,3 | 1,1 | 2,3 | 1,1 | 2,3 | 1,1 | 2,3 | 1,1 | 3,2 | 1,1 |
| 0.9 | 3,3 | 4,1 | 4,4 | 3,2 | 4,4 | 3,2 | 4,4 | 3,2 | 4,4 | 2,3 |
| | 1,4 | 2,2 | 2,3 | 1,1 | 2,3 | 1,1 | 2,3 | 1,1 | 3,2 | 1,1 |

In Table A.2, each firm's profit obtained in each of the four possible combinations of strategies are ranked from lowest (rank of 1) to highest (rank of 4) for each combination of $\alpha_v$ and $\alpha_d$. The result is denoted by $(k_A, k_B)$ for $k_A, k_B = 1,..4$, where $k_A$ is the rank of preference of the strategy for Firm $a$ and $k_B$ is the rank of preference of the strategy for Firm $b$, according to the payoff matrix in Section 5.4.

Table A.3: Firms' optimal revenue in Scenario S1 for combinations of $\alpha_v$ & $\alpha_d$

| $\alpha_v$ \ $\alpha_d$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.1 | 1.064 | 1.041 | 1.029 | 1.022 | 1.017 |
| 0.2 | 1.143 | 1.110 | 1.073 | 1.045 | 1.036 |
| 0.3 | 1.274 | 1.209 | 1.159 | 1.126 | 1.104 |
| 0.4 | 1.423 | 1.330 | 1.254 | 1.204 | 1.168 |
| 0.5 | 1.570 | 1.433 | 1.383 | 1.312 | 1.261 |
| 0.6 | 1.699 | 1.522 | 1.453 | 1.369 | 1.308 |
| 0.7 | 1.742 | 1.563 | 1.479 | 1.398 | 1.336 |
| 0.8 | 1.684 | 1.531 | 1.432 | 1.388 | 1.336 |
| 0.9 | 1.592 | 1.464 | 1.390 | 1.341 | 1.307 |

Table A.4: Firms' optimal cost in Scenario S1 for combinations of $\alpha_v$ & $\alpha_d$

| $\alpha_v$ \ $\alpha_d$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.1 | 0.095 | 0.093 | 0.092 | 0.092 | 0.091 |
| 0.2 | 0.113 | 0.136 | 0.117 | 0.093 | 0.093 |
| 0.3 | 0.178 | 0.196 | 0.187 | 0.181 | 0.178 |
| 0.4 | 0.230 | 0.256 | 0.240 | 0.230 | 0.223 |
| 0.5 | 0.253 | 0.273 | 0.325 | 0.307 | 0.294 |
| 0.6 | 0.272 | 0.287 | 0.337 | 0.316 | 0.301 |
| 0.7 | 0.274 | 0.290 | 0.335 | 0.315 | 0.300 |
| 0.8 | 0.224 | 0.276 | 0.257 | 0.303 | 0.291 |
| 0.9 | 0.182 | 0.217 | 0.242 | 0.281 | 0.274 |

Table A.5: Firms' optimal revenue in Scenario S3 for combinations of $\alpha_v$ & $\alpha_d$

| $\alpha_v$ \ $\alpha_d$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.1 | 1.003 | 1.003 | 1.003 | 1.003 | 1.003 |
| 0.2 | 1.004 | 1.003 | 1.003 | 1.003 | 1.003 |
| 0.3 | 1.004 | 1.004 | 1.004 | 1.003 | 1.003 |
| 0.4 | 1.004 | 1.004 | 1.004 | 1.004 | 1.004 |
| 0.5 | 1.005 | 1.005 | 1.004 | 1.004 | 1.004 |
| 0.6 | 1.005 | 1.005 | 1.005 | 1.004 | 1.004 |
| 0.7 | 1.006 | 1.005 | 1.005 | 1.005 | 1.004 |
| 0.8 | 1.031 | 1.006 | 1.005 | 1.005 | 1.005 |
| 0.9 | 1.152 | 1.006 | 1.006 | 1.005 | 1.005 |

Table A.6: Firms' optimal cost in Scenario S3 for combinations of $\alpha_v$ & $\alpha_d$

| $\alpha_v$ \ $\alpha_d$ | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 |
|---|---|---|---|---|---|
| 0.1 | 6.6500E-03 | 6.6495E-03 | 6.6490E-03 | 6.6485E-03 | 6.6482E-03 |
| 0.2 | 6.6593E-03 | 6.6581E-03 | 6.6571E-03 | 6.6563E-03 | 6.6555E-03 |
| 0.3 | 6.6686E-03 | 6.6668E-03 | 6.6653E-03 | 6.6640E-03 | 6.6629E-03 |
| 0.4 | 6.6778E-03 | 6.6755E-03 | 6.6735E-03 | 6.6718E-03 | 6.6703E-03 |
| 0.5 | 6.6871E-03 | 6.6842E-03 | 6.6817E-03 | 6.6795E-03 | 6.6777E-03 |
| 0.6 | 6.6965E-03 | 6.6929E-03 | 6.6899E-03 | 6.6873E-03 | 6.6851E-03 |
| 0.7 | 6.7058E-03 | 6.7017E-03 | 6.6981E-03 | 6.6951E-03 | 6.6925E-03 |
| 0.8 | 3.1396E-02 | 6.7104E-03 | 6.7064E-03 | 6.7029E-03 | 6.6999E-03 |
| 0.9 | 1.5454E-01 | 6.7192E-03 | 6.7147E-03 | 6.7108E-03 | 6.7074E-03 |

**Appendix B**

**In each scenario, and for each possible combination of the firm's reward periods decisions $(N_a, N_b)$, we identify the set of choices that are available to consumers (i.e., all combinations of $(n_a, n_b)$). Then, we calculate the expected number of customers ($\tau$) for each of these choices.**

**Example: scenario S1 (both firms restrict reward redemption)**

**For $N_a = 1, N_b = 2$, there are 6 different combinations of possible consumer choices: (0,0), (0,1), (0,2), (1,0), (1,1), and (1,2). We calculate the expected number of customers who would choose each of these alternatives and obtain the following system of nonlinear equations:**

$$\begin{cases} \tau(0,0) = \tau(0,2) * Q_{B1}(0,2) + \tau(1,0) * Q_{A1}(1,0) \\ \tau(0,1) = \tau(0,0) * Q_{B0}(0,0) + \tau(1,1) * Q_{A1}(1,1) \\ \tau(0,2) = \tau(0,1) * Q_{B0}(0,1) + \tau(1,2) * Q_{A1}(1,2) \\ \tau(1,0) = \tau(0,0) * Q_{A0}(0,0) + \tau(1,2) * Q_{B1}(1,2) \\ \tau(1,1) = \tau(0,1) * Q_{A0}(0,1) + \tau(1,0) * Q_{B0}(1,0) \\ \tau(1,2) = \tau(0,2) * Q_{A0}(0,2) + \tau(1,1) * Q_{B0}(1,1) \end{cases}$$

**Where:**

$$Q_z(n_a, n_b) = \exp(D_z) / \big(\exp(D_{A0}) + \exp(D_{A1}) + \exp(D_{B0}) + \exp(D_{B1})\big), z \in \{A0, A1, B0, B1\}$$

**and $D_z$ are given in equations 3-6 for S1.**

**Solving the above system of equations in $\tau(0,0)$, $\tau(0,1)$, $\tau(0,2)$, $\tau(1,0)$, $\tau(1,1)$, and $\tau(1,2)$, we can obtain the expected number of customers for each possible combination of $(n_a, n_b)$, when Firm $a$ and Firm $b$ choose the reward periods $N_a = 1, N_b = 2$. Then, we inject these expressions in equations 7-13 to get the firms' profit functions in this case.**

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